Dimension estimates for certain sets of infinite complex continued fractions

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Abstract

We prove upper and lower estimates on the Hausdorff dimension of sets of infinite complex continued fractions with finitely many prescribed Gaussian integers. Especially we will conclude that the dimension of theses sets is not zero or two and there are such sets with dimension greater than one and smaller than one.
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1 Introduction

Continued fractions were studied in number theory since the work of Wallis in the 17th century, see [7]. The first dimensional theoretical perspective on infinite real continued fractions can be found in the work of Jarnik [12], who introduced upper and lower estimates on the Hausdorff dimension of sets of continued fractions with bounded digits. The problem of calculating the dimension of these sets has been addressed by several authors [6, 1, 2, 8, 9]. In a resent work Jenkinson and Pollicott provide a fast algorithm to approximate this dimension [13].
Dimension theoretical aspects of infinite complex continued fractions were studied by Gardner, Mauldin and Urbanski [5, 16]. They proofed that the set of complex continued fractions with arbitrary Gaussian integers from $\mathbb{N} + \mathbb{Z}i$ has Hausdorff dimension greater than one and smaller than two.
We consider here infinite complex continued fractions and ask for the Hausdorff dimension of the set of continued fractions with digits coming from a finite set $A \subset \mathbb{N}[i]$. Using the Moran formula from the theory of iterated function systems [15] we are able to give upper and lower estimates on the Hausdorff dimension of these sets, see theorem 2.1. We will show that the dimension of the sets is not zero or two and there are such sets with dimension grater than one and smaller than one, see corollary 2.1 and 2.2. In addition we provide explicit estimates in selected examples.

2 Notations, Results and Examples

Given a sequence $z_n \in \mathbb{C}$ for $n \in \mathbb{N}_0$ of Gaussian integers we define the infinite complex continued fraction by

$$[z_0; z_1, z_2, \ldots] := z_0 + \frac{1}{1 + \frac{z_1}{1+z_2\ldots}} \in \mathbb{C}.$$

It is well known that every complex number can be represented as an infinite continued fractions of Gaussian integers using the Hurwitz algorithm [10]. Now fix a finite set

$$A = \{a_j + b_j i \mid a_j, b_j \in \mathbb{N}, \quad j = 1, \ldots, m\} \subset \mathbb{N}[i].$$

We consider the set of all infinite continued fractions having fractional entries coming from A,

$$\mathfrak{C}(A) := \{[0; z_1, z_2, \ldots] \mid z_n \in A, \quad n \in \mathbb{N}\} \subset \mathbb{C}.$$

Obviously the set $\mathfrak{C}(A)$ is uncountable and it is a null set with respect to the two-dimension Lebesgue measure (this is immediate from corollary 2.1 below). Thus we are interested in the Hausdorff dimension of this set. Recall [3, 17] that the d-dimensional Hausdorff measure of a set $\mathfrak{C} \subseteq \mathbb{C}$ is

$$H^d(\mathfrak{C}) = \lim_{\epsilon \longmapsto 0} \inf\{\sum_{i=1}^{\infty} \text{diameter}(C_i)^d | \mathfrak{C} \subseteq \bigcup_{i=1}^{\infty} C_i,\ \text{diameter}(C_i) < \epsilon\}.$$

The Hausdorff dimension of $\mathfrak{C}$ is given by

$$\dim_H \mathfrak{C} = \sup\{d | H^d(\mathfrak{C}) = \infty\} = \inf\{d | H^d(\mathfrak{C}) = 0\}.$$

Now we are able to sate our main result on $\dim_H \mathfrak{C}(A)$.

Theorem 2.1 *For a finite set $A \subset \mathbb{N}[i]$ let $D, d \in \mathbb{R}^+$ be the unique real numbers fulfilling*

$$\sum_{a+bi \in A} (\frac{1}{a^2 + b^2})^D = 1$$

$$\sum_{a+bi \in A} (\frac{1}{a^2 + b^2 + (1+\sqrt{2}) \max\{a, b\} + 1})^d = 1.$$

We have

$$d \le \dim_H \mathfrak{C}(A) \le D.$$

With an additional argument this theorem as the following corollary:

Corollary 2.1 *For all finite sets $A \subset \mathbb{N}[i]$ we have $\dim_H \mathfrak{C}(A) < 2$; on the other hand $\dim_H \mathfrak{C}(A) > 0$ if A has more than one element.*

Proof.

$$\sum_{a+bi\in A}(\frac{1}{a^2+b^2})^2 \le \sum_{a=1}^{\infty}\sum_{b=1}^{\infty}(\frac{1}{a^2+b^2})^2 \le \frac{1}{4}+\sum_{k=2}^{\infty}(2k-1)(\frac{1}{k^2+1})^2$$

$$\le -\frac{3}{4}+\sum_{k=1}^{\infty}\frac{2k-1}{k^4} = -3/4+2\zeta(3)-\zeta(4) < 1.$$

Hence $D < 2$. If A has more than one element we have

$$\sum_{a+bi\in A}(\frac{1}{a^2+b^2+(1+\sqrt{2})\max\{a,b\}+1})^0 > 1,$$

hence $d > 0$. The result now follows from our theorem. □

By a similar argument we get the second corollary.

Corollary 2.2 *There exists finite sets $A \subset \mathbb{N}[i]$ with $\dim_H \mathfrak{C}(A) > 1$ and there exists such sets with $\dim_H \mathfrak{C}(A) < 1$.*

Proof.

$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}(\frac{1}{a^2+b^2+(1+\sqrt{2})\max\{a,b\}+1}) \ge \sum_{k=1}^{\infty}\frac{2k-1}{2k^2+(1+\sqrt{2})k+1}$$

$$\ge \sum_{k=1}^{\infty}(\frac{2}{(4+\sqrt{2})k}-\frac{1}{2k^2}) = \infty$$

Hence for a suitable choice of A we have

$$\sum_{a+bi\in A}\frac{1}{a^2+b^2+(1+\sqrt{2})\max\{a,b\}+1} > 1.$$

For this set A we have $d > 1$. On the other hand consider $A = \{1+i, 2+i\}$. We have

$$\sum_{a+bi\in A}\frac{1}{a^2+b^2} < 1,$$

hence $D < 1$. The result again follows from our theorem. □

We remark that it is possible deduce the last corollaries from theorem 1 and theorem 2 of [5] by a few additional arguments. To obtain these results from our main theorem seems to us more transparent.

Our last corollary gives the obvious explicit upper and lower bounds following from theorem 2.1:

Corollary 2.3 *For a finite set $A \subset \mathbb{N}[i]$ with cardinality $|A|$ we have,*

$$\frac{\log(|A|)}{\max_A \log(a^2+b^2)} \le \dim_H \mathfrak{C}(A) \le \frac{\log(|A|)}{\min_A \log(a^2+b^2+(1+\sqrt{2})\max\{a,b\}+1)}.$$

The estimates in this corollary are of course very crude. At the end of this section we will apply theorem 2.1 directly to a few examples. Let $A = \{2+2i, 3+2i, 3+3i\}$. The numbers D and d are given by

$$(\frac{1}{8})^D + (\frac{1}{13})^D + (\frac{1}{18})^D = 1 \text{ and } (\frac{1}{11+2\sqrt{2}})^d + (\frac{1}{17+3\sqrt{2}})^d + (\frac{1}{22+3\sqrt{2}})^d = 1,$$

which implies $0.36 < \dim_H \mathfrak{C}(A) < 0.44$, which is an acceptable estimate. If we consider values with small modulus $A = \{1+i, 2+i, 1+2i\}$ we get

$$(\frac{1}{2})^D + 2(\frac{1}{5})^D = 1 \text{ and } (\frac{1}{4+\sqrt{2}})^d + 2(\frac{1}{8+2\sqrt{2}})^d = 1.$$

This gives $0.49 < \dim_H \mathfrak{C}(A) < 0.91$, which is not very good. Let us consider one more example $A = \{3+i, 2+4i\}$. We get

$$(\frac{1}{10})^D + (\frac{1}{20})^D = 1 \text{ and } (\frac{1}{14+3\sqrt{2}})^d + (\frac{1}{25+4\sqrt{2}})^d = 1$$

and thus $0.21 < \dim_H \mathfrak{C}(A) < 0.27$. As a last example consider $A = \{a+bi | a, b = 1 \dots 4\}$. An elementary calculation shows that theorem 2.1 gives $1 < \dim_H \mathfrak{C}(A) < 1.33$. We like to remark here that it is possible to find an algorithm using thermodynamic formalism that approximate the dimension of $\mathfrak{C}(A)$. We could apply the recent approach of Jekinsion and Policott [14] to infinite complex continues fractions. This approach has the disadvantage that it is not possible to perform necessary calculations without using a computer, which would change the field of our research to computational mathematics.

3 Proof of the result

For $(a,b) \in \mathbb{N}^2$ consider transformations $T_{a,b} : \mathbb{C} \longmapsto \mathbb{C}$ given by

$$T_{a,b}(z) = \frac{1}{z+a+bi}$$

We need three elementary lemmas concerning these transformations to apply the dimension theory of iterated functions systems to the set $\mathfrak{C}(A)$. First we restrict the maps to the open ball $B_{1/2}(1/2) = \{z \in \mathbb{C} \mid |z - 1/2| < 1/2\}$.

Lemma 3.1 *For $(a,b) \in \mathbb{N}^2$ we have*

$$T_{a,b}(B_{1/2}(1/2)) \subset B_{1/2}(1/2).$$

Proof. For $I(z) = 1/z$ we have

$$I(B_r(z)) = B_{r/(|z|^2-r^2)}(\frac{\bar{z}}{|z|^2-r^2}),$$

if $|z| \neq r$. Applying the translation with $a+bi$ we obtain

$$T_{a,b}(B_r(z)) = B_{r/(|z+a+bi|^2-r^2)}\left(\frac{\overline{z+a+bi}}{|z+a+bi|^2-r^2}\right)$$

if $|z+a+bi| \neq r$. Especially we get

$$T_{a,b}(B_{1/2}(1/2)) = B_{\frac{1}{2a+2a^2+2b^2}}\left(\frac{1/2+a-bi}{a+a^2+b^2}\right).$$

since $|1/2+a+bi| > 1/2$ for $a, b > 0$. We have to show the distance of the center of the image to $1/2$ plus the radius of the image is less or equal to $1/2$. This means

$$|\frac{1/2+a-bi}{a+a^2+b^2} - \frac{1}{2}| + \frac{1}{2a+2a^2+2b^2} \leq \frac{1}{2}$$
$$\Leftrightarrow |a^2+b^2-a-1+2bi|^2 \leq (a^2+b^2+a-1)^2$$
$$\Leftrightarrow (a^2+b^2-a-1)^2 + 4b^2 \leq (a^2+b^2+a-1)^2$$
$$\Leftrightarrow 4a+4b^2 \leq 4(a^2+b^2)a \Leftrightarrow 1 \leq a^2,$$

which is obviously true for a $a \in \mathbb{N}$. $\square$

Next we show that the images of the open balls $B_{1/2}(1/2)$ under different $T_{a,b}$ are disjoint.

Lemma 3.2 *If* $(a_1, b_1) \neq (a_2, b_2)$ *we have*

$$T_{a_1,b_1}(B_{1/2}(1/2)) \cap T_{a_2,b_2}(B_{1/2}(1/2)) = \emptyset.$$

Proof. We have to show that the distance of the balls at hand is bigger or equal to the sum of there radii, this is

$$|\frac{1/2+a_1-b_1i}{a_1+a_1^2+b_1^2} - \frac{1/2+a_2-b_2i}{a_2+a_2^2+b_2^2}| \geq \frac{1}{2a_1+2a_1^2+2b_1^2} + \frac{1}{2a_2+2a_2^2+2b_2^2}.$$

With $d_1 = a_1+a_1^2+b_1^2$ and $d_2 = a_2+a_2^2+b_2^2$ we have to show

$$|\frac{1/2+a_1-b_1i}{d_1} - \frac{1/2+a_2-b_2i}{d_2}| \geq \frac{1}{2d_1} + \frac{1}{2d_2}$$
$$\Leftrightarrow |d_2(1/2+a_1-b_1i) - d_1(1/2+a_2-b_2i)|^2 \geq (d_1/2+d_2/2)^2$$
$$\Leftrightarrow ((1/2+a_1)d_2 - (1/2+a_2)d_1)^2 + (b_1d_2-b_2d_1)^2 \geq (d_1/2+d_2/2)^2$$
$$\Leftrightarrow d_1d_2^2 - (1/2+a_1+a_2+2a_1a_2)d_1d_2 + d_2d_1^2 - 2b_1d_2b_2d_1 \geq d_1d_2/2$$
$$\Leftrightarrow d_1+d_2-(1+a_1+a_2+2a_1a_2+2b_1b_2) \geq 0$$
$$\Leftrightarrow (a_1-a_2)^2 + (b_1-b_2)^2 - 1 \geq 0.$$

This is obviously true under our assumption. $\square$

The last lemma contains estimates on the modulus of derivative of the maps on the closed ball $\bar{B}_{1/2}(1/2)$.

Lemma 3.3 *For $(a,b) \in \mathbb{N}^2$ we have*

$$\max\{|T'_{a,b}(z)| z \in \bar{B}_{1/2}(1/2))\} \leq \frac{1}{a^2+b^2},$$

$$\min\{|T'_{a,b}(z)| z \in \bar{B}_{1/2}(1/2))\} \geq \frac{1}{a^2+b^2+(1+\sqrt{2})\max\{a,b\}+1}.$$

Proof. For $z \in \bar{B}_{1/2}(1/2)$ we have

$$T'_{a,b}(z) = \frac{-1}{(z+a+bi)^2} \quad \text{and hence} \quad |T'_{a,b}(x+yi)| = \frac{1}{(x+a)^2+(y+b)^2}.$$

Now the first estimate is obvious. For the second part note that

$$\max\{(x+a)^2+(y+b)^2 | z = x+iy \in \bar{B}_{1/2}(1/2))\}$$

$$= \max\{(x+a)^2+(y+b)^2 | (x-1/2)^2+y^2 \leq 1/4,\ x,y \in \mathbb{R}\}$$

$$= a^2+b^2+\max\{x+2xa+2yb | y^2 \leq x-x^2, x \in [0,1], y \in [-1/2,1/2]\}$$

$$\leq 1+a^2+b^2+2\max\{xa+yb | y^2 \leq x-x^2, x \in [0,1], y \in [-1/2,1/2]\}$$

$$\leq 1+a^2+b^2+2\max\{a,b\}\max\{x+y | y^2 \leq x-x^2, x \in [0,1], y \in [-1/2,1/2]\}$$

$$\leq 1+a^2+b^2+(1+\sqrt{2})\max\{a,b\}$$

using Lagrange in the last estimate. This implies the result. □

Given a set finite $A \subset \mathbb{N}[i]$ consider the iterated function system (IFS) in the sense of Hutchinson [11]:

$$(\bar{B}_{1/2}(1/2), \{T_{a,b} | a+bi \in A\}).$$

By lemma 3.1 this IFS is well defined with attractor $\mathfrak{C}(A)$; i.e.

$$\mathfrak{C}(A) = \bigcup_{a+bi \in A} T_{a,b}(\mathfrak{C}(A)).$$

By lemma 3.2 the IFS fulfills the open set condition, first introduced by Moran [15]. Moreover by lemma 3.3 we have

$$\frac{|z_1-z_2|}{a^2+b^2+(1+\sqrt{2})\max\{a,b\}+1} \leq |T_{a,b}(z_1)-T_{a,b}(z_2)| \leq \frac{|z_1-z_2|}{a^2+b^2}$$

for all $z_1, z_2 \in \bar{B}_{1/2}(1/2)$ and all $a+bi \in A$. Now theorem 2.1 is a direct application of theorem 8.8 of Falconer [4], a well know result in the dimension theory of IFS, which goes back to Moran [15].

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