Fractal attractors induced by β-shifts

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Abstract

We describe a class of fractal attractors induced by β-shifts. We use a coding by these shifts to show that the systems are mixing with topological entropy $\log\beta$ and have an ergodic measure of full entropy. Moreover we determine the Hausdorff dimension of the attractor.

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1 Introduction

Fractal attractors are a central subject in the modern theory of dynamical systems. Famous examples coming from applied mathematics are the Lorenz attractor, the Hennon attractor, the Rössler attractor and the Ikeda attractor, see [6] for instance. Well known examples, that are of importance from a theoretical perspective, are solenoidal attractors [2, 9] and attractors of generalized Bakers maps [1, 8] .

We introduce here a new class of fractal attractors that are induced by β-shifts. β-shifts are intensively studied in arithmetic resp. symbolic dynamics, see [13] for an overview. They describe the dynamics of the expanding map $f(x) = \beta x \bmod 1$ on the Intervall $[0, 1)$, where $\beta > 1$.

For parameters $\beta \in (1, 2)$ and $\tau \in (0, 0.5)$ let us consider the map $f : [0, 1]^2 \to [0, 1]^2$ given by

$$f(x,y) = \begin{cases} (\beta x, \tau y), & x \in [0, \beta^{-1}] \\ (\beta x - 1, \tau y + (1-\tau)), & x \in (\beta^{-1}, 1] \end{cases} .$$

We define the compact attractor of the dynamical system $([0, 1]^2, f)$ by

$$\Lambda_{\beta,\tau} = \text{closure}(\bigcap_{i=0}^{\infty} f^i([0,1]^2)).$$

In the overlapping case $\tau \in (0.5, 1)$ this attractor was studied in [4], especially the question if there is an absolutely continuous ergodic measure for the system is addressed. We consider here the non-overlapping case $\tau \in (0, 0.5)$.

In the next section we will describe the dynamics of f on the attractor $\Lambda_{\beta,\tau}$ symbolically

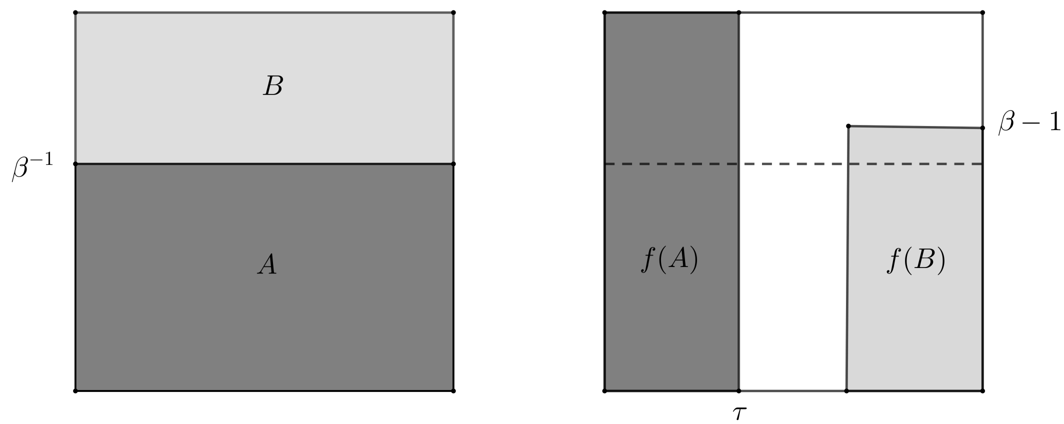

Figure 1: The action of f on $[0,1]^2$

using β-shifts. With the help of this description we will show that the system $(\Lambda_{\beta,\tau}, f)$ is topological mixing. In section three we study the entropy of the dynamical system. Using the symbolic coding we show that topological entropy of the system is $\log(\beta)$ and that there is an ergodic measure of full entropy. In the last section we determine the Hausdorff dimension of the attractor $\Lambda_{\beta,\tau}$ which turn to be in $(1,2)$ for all $\beta \in (1,2)$ and $\tau \in (0, 0.5)$. This means that the attractor is in fact a fractal according to the usual definition.

2 Symbolic dynamics

Consider the space of bi-infinite sequences on two symbols $\Sigma = \{0,1\}^{\mathbb{Z}}$ with the natural product topology which is induced by the metric

$$d((s_k),(t_k)) = \sum_{i=0}^{\infty} |s_k - t_k| 2^{-|k|}.$$

The shift $\sigma : \Sigma \to \Sigma$ given by $\sigma((s_k)) = (s_{k-1})$ is an universal model in chaotic dynamics, see [5] for instance. For a real number $\beta \in (1,2)$ we consider here a subshift given by

$$X_\beta = \{(s_k) \in \Sigma \mid \sum_{k=1}^{\infty} s_{i-k}\beta^{-k} < 1 \text{ for all } i \in \mathbb{Z}\}$$

and its closure

$$\overline{X}_\beta = \{(s_k) \in \Sigma \mid \sum_{k=1}^{\infty} s_{i-k}\beta^{-k} \leq 1 \text{ for all } i \in \mathbb{Z}\}.$$

In addition we use

$$\overline{X}_\beta^\star = \overline{X}_\beta \backslash \{(s_k) | \exists i \in \mathbb{Z} : s_{i-k} = 0 \text{ for all } k \in \mathbb{Z}\}.$$

The sets X_β, $\overline{X}_\beta$ and $\overline{X}_\beta^\star$ are obviously forward and backward invariant under the shift maps σ. The dynamical systems $(\overline{X}_\beta, \sigma)$ are know as two-sided β-shift.
Now we introduce a coding map $\pi : \overline{X}_\beta \to Q$ by

$$\pi((s_k)) = (\sum_{k=1}^{\infty} s_{-k}\beta^{-k}, \sum_{k=0}^{\infty} s_k(1-\tau)\tau^k).$$

This map has the following properties:

Proposition 2.1 *π is continuous with $\pi(\overline{X_\beta}) = \Lambda_{\beta,\tau}$ and the map conjugates f and the shift σ on $\overline{X}_\beta^\star$, that means*

$$f(\pi((s_k))) = \pi(\sigma((s_k)))$$

for all sequences in $(s_k) \in \overline{X_\beta}^\star$. Moreover f is injective on X_β.

Proof. Let $(s_k^{(i)})$ be a sequence of sequences in $\overline{X_\beta}$ with $\lim_{i\to\infty}(s_k^{(i)}) = (s_k)$. Let

$$M(i) = \max\{n \mid s_k^{(i)} = s_k,\ k \in \{-n, \ldots, -1, 0, 1, \ldots, n\}\}$$

By the definition of the metric on $\overline{X_\beta}$ we get $\lim_{i\to\infty} M(i) = \infty$. Looking at the definition of the coding map π this obviously implies $\lim_{i\to\infty} \pi((s_k^{(i)})) = \pi((s_k))$. Hence π is continuous. We now show the conjugacy. Consider a sequence $(s_k) \in \overline{X_\beta}^\star$. If $\sum_{k=1}^\infty s_{-k}\beta^{-k} \le \beta^{-1}$ we have $s_{-1} = 0$ ($s_{-1} = 1$ would imply $s_{-k} = 0$ for all $k \ge 2$). Hence we get

$$f(\pi((s_k))) = (\beta \sum_{k=1}^{\infty} s_{-k}\beta^{-k}, \tau \sum_{k=0}^{\infty} s_k(1-\tau)\tau^k)$$

$$= (\sum_{k=1}^{\infty} s_{-k-1}\beta^{-k}, \sum_{k=0}^{\infty} s_{k-1}(1-\tau)\tau^k) = \pi(\sigma((s_k))).$$

If $\sum_{k=1}^\infty s_{-k}\beta^{-k} > \beta^{-1}$ we have $s_{-1} = 1$, an thus

$$f(\pi((s_k))) = (\beta \sum_{k=1}^{\infty} s_{-k}\beta^{-k} - 1, \tau \sum_{k=0}^{\infty} s_k(1-\tau)\tau^k + (1-\tau))$$

$$= (\sum_{k=1}^{\infty} s_{-k-1}\beta^{-k}, \sum_{k=0}^{\infty} s_{k-1}(1-\tau)\tau^k) = \pi(\sigma((s_k))).$$

Now consider the map f on X_β and let $(s_k), (t_k) \in X_\beta$. By the definition of X_β

$$\sum_{k=1}^{\infty} s_{-k}\beta^{-k} = \sum_{k=1}^{\infty} t_{-k}\beta^{-k}$$

implies $s_{-k} = t_{-k}$ for all $k \in \mathbb{N}$. Since $\tau < 1/2$

$$\sum_{k=0}^{\infty} s_k(1-\tau)\tau^k = \sum_{k=0}^{\infty} t_k(1-\tau)\tau^k$$

implies $s_k = t_k$ for all $k \in \mathbb{N}_0$. Hence f is injective on X_β. We now define

$$S_\beta = \{(s_k) \in \Sigma \mid \sum_{k=1}^{\infty} s_{i-k}\beta^{-k} \leq 1 \text{ for all } i \leq 0\} \backslash \{(s_k) | \exists i \in \mathbb{N} : s_{i-k} = 0 \text{ for all } k \in \mathbb{Z}\}.$$

Note that $\bigcap_{i=0}^{\infty} \sigma^i(S_\beta) = \overline{X}_\beta^\star$. Let

$$C_\tau = \{\sum_{k=0}^{\infty} s_k(1-\tau)\tau^k \mid s_k \in \{0,1\}, k \in \mathbb{N}_0\}$$

and

$$I_\beta = [0,1] \backslash \{\sum_{k=1}^{n} s_k \beta^{-k} \mid s_k \in \{0,1\}, k = 1, \ldots, n\}$$

We have

$$I_\beta \times C \subseteq \pi(S_\beta) \subseteq [0,1] \times C.$$

Since $f(\pi(S_\beta)) = \pi(\sigma(S_\beta)$ we get

$$f^n(I_\beta \times C) \subseteq \pi(\sigma^n(S_\beta)) \subseteq f^n([0,1] \times C)$$

and hence

$$\bigcap_{n=0}^{\infty} f^n(I_\beta \times C) \subseteq \pi(\overline{X}_\beta^\star) \subseteq \bigcap_{n=0}^{\infty} f^n([0,1] \times C).$$

The closure of I_β is $[0,1]$ and the closure of $\overline{X}_\beta^\star$ is $\overline{X}_\beta$. Thus we obtain $\pi(\overline{X_\beta}) = \Lambda_{\beta,\tau}$. $\square$

Using this proposition we get the following result on the dynamics of f on the attractor $\Lambda_{\beta,\tau}$.

Theorem 2.1 *The dynamical system $(\Lambda_{\beta,\tau}, f)$ is topological mixing: For nonempty open sets A and B there exists an integer N such that, for all $n > N$*

$$f^n(A) \cap B \neq \emptyset.$$

Proof. It is known that the system $(\overline{X_\beta}, \sigma)$ is topological mixing, see [12]. Let A and B be two open sets in $\Lambda_{\beta,\tau}$. By continuity of π the preimages $\pi^{-1}(A)$ and $\pi^{-1}(B)$ are open and they are contained in $\overline{X_\beta}$. Hence there ist an N, such that for all $n > N$ there exists a sequence $s(n) \in \overline{X}_\beta$, such that

$$s(n) \in \sigma^n(\pi^{-1}(A)) \cap \pi^{-1}(B).$$

Since the set $\sigma^n(\pi^{-1}(A)) \cap \pi^{-1}(B)$ is open in $\overline{X}_\beta$ we may assume that $s(n) \in \overline{X}^\star_\beta$. Hence we get:

$$\begin{aligned}\pi(s(n)) &\in \pi(\sigma^n(\pi^{-1}(A)) \cap \pi^{-1}(B)) \subseteq \pi(\sigma^n(\pi^{-1}(A))) \cap \pi(\pi^{-1}(B)) \\ &= f^n(\pi((\pi^{-1}(A))) \cap \pi(\pi^{-1}(B)) \subseteq f^n(A) \cap B \neq \emptyset.\end{aligned}$$

Thus $(\Lambda_{\beta,\tau}, f)$ is topological mixing. □

3 Entropy

For convenience we first recall the definition of the topological and metric entropy. Let (X, f) be dynamical system on a compact space X. The entropy of an open covering $\mathfrak{U}$ of X is given $H(\mathfrak{U}) = \log \sharp\mathfrak{U}$, where $\sharp\mathfrak{U}$ is the minimal number of elements in $\mathfrak{U}$ that cover X. The entropy of the system (X, f) with respect to $\mathfrak{U}$ is

$$h(f, \mathfrak{U}) = \lim_{n\to\infty} \frac{1}{n} H(\mathfrak{U} \vee f^{-1}(\mathfrak{U}) \vee \cdots \vee f^{-n}(\mathfrak{U})),$$

where a covering $\mathfrak{U}_1 \vee \mathfrak{U}_2$ consists of the intersections of elements in $\mathfrak{U}_1$ and $\mathfrak{U}_2$. The topological entropy of the system is

$$h(f) = \sup\{h(f, \mathfrak{U}) \mid \mathfrak{U} \text{ is an open covering of } X\}.$$

A Borel probability measure μ on X is ergodic with respect to f if it is invariant, $\mu \circ f^{-1} = \mu$, and sets B with $f^{-1}(B) = B$ have measure zero or one. The metric entropy of a system (X, f, μ) with respect to a measurable partition $\mathfrak{P}$ is

$$h(f, \mu, \mathfrak{P}) = \lim_{n\to\infty} \frac{1}{n} H(\mu, \mathfrak{P} \vee f^{-1}(\mathfrak{P}) \vee \cdots \vee f^{-n}(\mathfrak{P})),$$

where is entropy of a measurable partition is

$$H(\mu, \mathfrak{P}) = -\sum_{P\in\mathfrak{P}} \mu(P) \log(\mu(P)).$$

The metric entropy of the system is

$$h(f, \mu) = \sup\{h(f, \mathfrak{P}) \mid \mathfrak{P} \text{ is a measurable partition of } X\}.$$

We recommend [14] and [5] for an introduction to entropy theory.

The entropy of β-shifts is well studied. The topological entropy of $(\overline{X_\beta}, \sigma)$ is given by $\log \beta$, $h(\sigma_{|\overline{X_\beta}}) = \log \beta$, see [12]. Moreover there is a shift ergodic measure μ_β on $\overline{X_\beta}$ with full entropy $h(\mu_\beta, \sigma) = \log \beta$, see [10]. We will transfer these results to the dynamical system $(\Lambda_{\beta,\tau}, f)$ using the symbolic coding in proposition 2.1. and prove:

Theorem 3.1 *The dynamical system* $(\Lambda_{\beta,\tau}, f)$ *has topological entropy* $\log(\beta)$ *and there is an ergodic measure of full entropy.*

Proof. Proposition 2.1. shows in the terminology of topological dynamics that $(\Lambda_{\beta,\tau}, f)$ is a factor of $(\overline{X_\beta}^\star, \sigma)$. It is well known and straightforward from the definition above, that this implies

$$h(f_{|\Lambda_{\beta,\tau}}) \le h(\sigma_{|\overline{X_\beta}^\star}) \le h(\sigma_{|\overline{X_\beta}}) = \log \beta.$$

Let μ be an ergodic measure for $(\overline{X_\beta}, \sigma)$. We show first that $\mu(\overline{X_\beta} \backslash \overline{X_\beta}^\star) = 0$. Let

$$A_k = \{(s_i) \mid s_k = 1,\ s_i = 0 \text{ for } i < k\}$$

for $k \in \mathbb{Z}$. The shifted sets $\sigma^i(A_k)$ and $\sigma^j(A_k)$ are disjoint for $i, j \in \mathbb{N}_0$ with $i \neq j$. Since μ ist σ-invariant this implies $\mu(A_k) = 0$, so $\mu(\{(s_k) | \exists i \in \mathbb{Z} : s_{i-k} = 0 \text{ for all } k \in \mathbb{Z}\}) = 0$. Now we prove $\mu(\overline{X_\beta} \backslash X_\beta) = 0$. We may decompose $\overline{X_\beta} \backslash X_\beta$ in the following way

$$\overline{X_\beta} \backslash X_\beta = \{(s_k) \in \Sigma \mid \text{For some } i \in \mathbb{Z} :\ \sum_{k=1}^{\infty} s_{i-k} \beta^{-k} = 1 \text{ and } s_k = 0,\ k \ge i\} = \bigcup_{i=-\infty}^{\infty} N_i,$$

where N_i contains all sequences $(s_k) \in \overline{X_\beta} \backslash X_\beta$ with $s_{i-1} = 1$ and $s_k = 0$ for $k \ge i$. Note that $\sigma^{-a}(N_i) \cap \sigma^{-b}(N_i) = \emptyset$ for $a \neq b$. Since $\mu(\sigma^{-a}(N_i)) = \mu(N_i)$ this implies $\mu(N_i) = 0$ and $\mu(\overline{X_\beta} \backslash X_\beta) = 0$.

Now we project μ to $\Lambda_{\beta,\tau}$ via $\nu = \pi(\mu) = \mu \circ \pi^{-1}$. By proportion 2.1 and the consideration above the measure space $(\overline{X_\beta}, \mu)$ and $(\Lambda_{\beta,\tau}, \nu)$ are measure theoretical isomorphic. Moreover the dynamical systems $(\overline{X_\beta}, f, \mu)$ and $(\Lambda_{\beta,\tau}, \sigma, \nu)$ are by proposition 2.1 measure theoretical conjugated with $f \circ \pi = \pi \circ \sigma$. It is well known and straightforward from the definition above, that this implies

$$h(f_{|\Lambda_{\beta,\tau}}, \nu) = h(\sigma_{|\overline{X_\beta}}, \mu).$$

If μ_β is the measure of full entropy for the β-shift the projected measure ν_β has full entropy for f,

$$h(f_{|\Lambda_{\beta,\tau}}, \nu_\beta) = \log \beta = h(f_{|\Lambda_{\beta,\tau}}).$$

Here we use the fact that the metric entropy is always bounded by topological entropy of a dynamical system. □

4 Dimension

In this section we determine the Hausdorff dimension of the attractor $\Lambda_{\beta,\tau}$ defined in section one. We refer to [3] or [11] for an introduction to dimension theory. Let us recall that the d-dimensional Hausdorff measure of a set $B \subseteq \mathbb{R}^2$ is given by

$$H^d(B) = \lim_{\epsilon \longmapsto 0} \inf\{\sum_{i=1}^{\infty} \text{diameter}(C_i)^d | B \subseteq \bigcup_{i=1}^{\infty} C_i,\ \text{diameter}(C_i) < \epsilon\}$$

and the Hausdorff dimension of B is

$$\dim_H B = \inf\{d \mid H^d(B) = 0\} = \sup\{d \mid H^d(B) = \infty\}.$$

As an upper bound on the Hausdorff dimension we will use the (lower) box-counting dimension:

$$\dim_H B \leq \underline{\lim}_{\epsilon\to 0} \frac{\log N_\epsilon(B)}{\log \epsilon^{-1}},$$

where $N_\epsilon(B)$ is the minimal number of squares with side length ϵ needed to cover B. As a lower bound we will use the Hausdorff dimension of a Borel probability measure μ, which is given by

$$\dim_H \mu = \inf\{\dim_H B \mid \mu(B) = 1\}.$$

We prove the following result:

Theorem 4.1 *For all $\beta \in (1,2)$ and $\tau \in (0, 0.5)$ we have*

$$\dim_H \Lambda_{\beta,\tau} = 1 + \frac{\log \beta}{\log \tau^{-1}}$$

Proof. If R is an aligned rectangle in $[-1,1]^2$, than $f(R)$ consists of one or two aligned rectangles. For simplicity we refer to a line segment here as a rectangle of side length zero. $f^n([-1,1]^2)$ consists of at most 2^n aligned rectangles $R_1, R_2, \ldots, R_t$. In the first coordinate direction f is an expansion with factor β hence $\sum_{i=1}^t x_i = \beta^n$, where x_i is the length of R_i in the first coordinate direction. In the second coordinate direction f is a contraction with factor τ. The length of R_i in the second coordinate direction is τ^n. The number of squares of side length τ^n, needed to cover R_i, is less than $(x_i/\tau^n + 1)$. Hence we have

$$N_{\tau^n}(\Lambda_{\beta,\tau}) \leq N_{\tau^n}(f^n([0,1]^2)) \leq \beta^n/\tau^n + t \leq \beta^n/\tau^n + 2^n \leq (\beta^n + 1)/\tau^n$$

and obtain

$$\dim_H \Lambda_{\beta,\tau} \leq \lim_{n\to\infty} \frac{\log N_{\tau^n}(\Lambda_{\beta,\tau})}{\log(\tau^{-n})} = \lim_{n\to\infty} \frac{\log((\beta^n+1)/\tau^n)}{\log(\tau^{-n})} = 1 + \frac{\log \beta}{\log \tau^{-1}}.$$

For a f-ergodic measure ν on $\Lambda_{\beta,\tau}$ we have the Ledrappier-Young formula for the dimension of the measure

$$\dim_H \nu = h(f,\nu)\left(\frac{1}{\log \beta} + \frac{1}{\log \tau^{-1}}\right),$$

see [7, 15]. The theory of Ledappier-Young is formulated for differentiable systems without singularity, but it may be applied in our context as well. The argument for this fact is given in [9], one has to guarantee existence of Lyapunov charts. Let ν_β now be the ergodic measure of full entropy for f described in the last section. We have

$$\dim_H \nu_\beta = 1 + \frac{\log \beta}{\log \tau^{-1}} \geq \dim_H \Lambda_{\beta,\tau},$$

which completes the proof. □

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