Dimension theory of representations of real (and complex) numbers

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b-adic representation

- Consider the b-adic representation of a real numbers $x \in [0,1]$:

$$x = \sum_{i=1}^{\infty} d_i(x) b^{-i}, \quad d_i(x) \in \{0, 1, \ldots, b-1\}.$$

- Choosing digits from $A \subseteq \{0, \ldots, b-1\}$ we define

$$\mathcal{D}_{\text{b-adic}}[A] := \{x \in [0,1] | d_i(x) \in A\}.$$

- If $2 < |A| < |B|$ the set $\mathcal{D}_{\text{b-adic}}[A]$ is uncountable and compact but of length zero and totally disconnected.

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Hausdorff dimension

- The d-dimension Hausdorff measure of $B \subseteq \mathbb{R}^n$ is

$$\mathfrak{H}^d(B) = \lim_{\epsilon \longmapsto 0} \inf\{\sum_{i=1}^{\infty} \mathrm{diam}(C_i)^d | B \subseteq \bigcup_{i=1}^{\infty} C_i, \ \mathrm{diam}(C_i) < \epsilon\}.$$

- The is a natural generalization of the n-dimensional Lebesgue measure to non-integer dimensions, $\mathfrak{L}^n = c_n \mathfrak{H}^n$.
- The Hausdorff dimension is given by

$$\dim B = \inf\{d \geq 0 | \mathfrak{H}^d(B) = 0\} = \sup\{d \geq 0 | \mathfrak{H}^d(B) = \infty\}$$

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Hausdorff (1919):

Theorem

$$\dim \mathcal{D}_{b\text{-}adic}[A] = \frac{\log |A|}{\log b}$$

- For the upper bound just cover the set by by $|A|^n$ intervals of length b^n.
- For the lower bound define a probability measure by $\mu(I_{a_1 a_2 \dots a_n}) = |A|^{-n}$. We have

$$x \in \mathcal{D} : \ \mu(B_r(x)) \leq c \ r^{\log |A| / \log b}.$$

By the mass distribution principle $\mathfrak{H}^{\log |A| / \log b}(\mathcal{D}) \geq 1/c$.

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Prescribed frequencies in *b*-adic representation

- Let $\mathbf{p} = (p_j)$ be a probability vector on $\{0, \dots b-1\}$. The entropy of $\mathbf{p}$ is
$$H(\mathbf{p}) = -\sum_{j=0}^{b-1} p_j \log p_j.$$
- Consider the set of real numbers in $[0,1]$ with given frequency of digits in the b-adic representation
$$\mathcal{F}_{\text{b-adic}}[\mathbf{p}] := \{x | \lim_{n \mapsto \infty} \frac{|\{i = 1, \dots n | d_i(x) = j\}|}{n} = p_j\}.$$
- $\mathcal{F}_{\text{b-adic}}[(1/b)]$ is the set of normal numbers to base b. **Borel (1909)**: Almost every number is normal.

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Theorem

$$\dim \mathcal{F}_{b\text{-adic}}[\mathbf{p}] = \frac{H(\mathbf{p})}{\log b} \; (=: \theta)$$

- Construct a measure $\mu(I_{d_1 d_2 \dots d_n}) = p_{d_1} p_{d_2} \cdots p_{d_n}$.
- $$x \in \mathcal{F}: \quad \lim_{n \longmapsto \infty} \frac{1}{n} \log \frac{\mu(I_{d_1 \dots d_n}(x))}{|I_{d_1 \dots d_n}(x)|^s} = -H(\mathbf{p}) + s \log(b)$$
- $s < \theta : \lim_{r \to \infty} \mu(B_r(x))/r^s = 0 \Rightarrow \mathfrak{H}^s(\mathcal{F}) = \infty$
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A modification of the dyadic representation

- Represent a real number $x \in (0,1]$ by a sequence in $\mathbb{N}^{\mathbb{N}}$:

$$x = \sum_{i=1}^{\infty} 2^{-(n_1(x)+\cdots+n_i(x))}, \quad n_i(x) \in \mathbb{N}.$$

- $n_i(x)$ is the distance between two digits 1 in the dyadic expansion.
- For $A \subseteq \mathbb{N}$ consider the set of real numbers $\mathcal{D}_{\text{m.dyadic}}[A]$ with digits in A.
- $\mathcal{D}_{\text{m.dyadic}}[\{1,2\}] = \{x | n_i(x) = 0 \Rightarrow n_{i+1}(x) = 1\}$ is called the golden Markov set.

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Theorem

The Hausdorff dimension d of $\mathcal{D}_{m.dyadic}[A]$ is d is given by

$$\sum_{i \in A} 2^{-id} = 1$$

- For $A = \{1, \ldots n\}$: $d = \log(s)/\log(2)$ where s is given by the solution $s \in (1,2)$ of $s^n - s^{n-1} \cdots - s - 1 = 0$.
- For the golden Markov set: $d = \log((\sqrt{5}+1)/2)/\log 2$.
- For $A = \{nj | n \in \mathbb{N}\}$ we have $d = 1/j$.
- For $A = \{nj + m | n \in \mathbb{N}_0\}$ d is given by $2^{-dj} + 2^{-dm} = 1$.

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Theorem

$$\dim \mathcal{F}_{m.dyadic}[\mathbf{p}] = \frac{H(\mathbf{p})}{E(\mathbf{p}) \log 2}$$

- For the equidistribution

$$\dim \mathcal{F}_{\text{m.dyadic}}[(1/n, \ldots, 1/n)] = \frac{2 \log(n)}{(n+1) \log(2)}$$

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Continued fraction representation

- Represent a real number $x \in (0,1)$ by a continued fraction:

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Jarnik (1929):

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As a corollary of Jarnik's dimension estimate:

$$\dim_H\{x \in (0,1) \mid (n_k(x)) \text{ is bounded}\} = 1$$

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$$\dim_H\{x \in (0,1) \mid \lim_{k\to\infty}(n_k(x)) = \infty\} = 1/2$$

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Complex continued fractions

- For $z \in \mathbb{C}$ consider the Hurwiz continued fraction

$$z = c_0 + \frac{1}{c_1 + \frac{1}{c_2 + \dots}}, \quad c_j = a_j + b_j i \in \mathbb{Z}[i]$$

- The digits c_j are given by

$$z_{j+1} = 1/z_j - [1/z_j] = 1/z_j - c_j$$

with $c_0 = [z]$ and $z_0 = z - c_0$ where $[.]$ denotes rounding to the nearest element of $\mathbb{Z}[i]$.

- For $A \subseteq \mathbb{N}[i]$ consider the set of Hurwitz continued fractions $\mathcal{D}_{\text{comlex}}[A]$ with digits in A.

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Estimating the modulus of the derivative of $Tz = 1/(z + a + bi)$ on the ball $B_{1/2}(1/2)$ one proves:

Theorem

$$d < \dim \mathcal{D}_{comlex}[A] < D$$

$$\sum_{a+bi \in A} \left(\frac{1}{a^2 + b^2}\right)^D = 1$$

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- $0.21 < \dim \mathcal{D}_{\text{comlex}}[\{3 + i, 2 + 4i\}] < 0.27$
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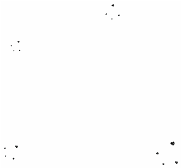

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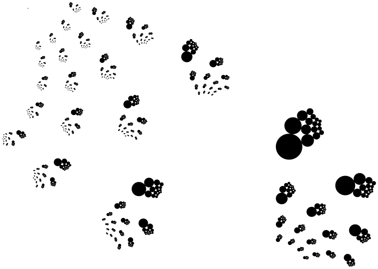

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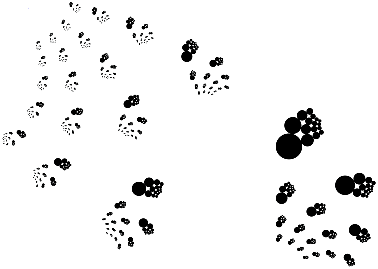

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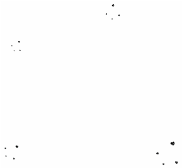

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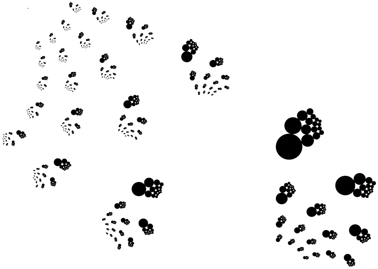

Continued logarithm representation

- Consider the continued logarithm representation to base $m \geq 3$ of $x \in [0,1]$:

$$x = \lim_{n\to\infty} \log_m(d_n(x) + \log_m(d_{n-1}(x) + \log_m(\cdots + \log_m(d_1(x)))$$

with digits in $\{1,\dots,m-1\}$.

- The representation is unique up to a countable set and in almost all numbers all digits appear.
- For $m \geq 4$ choosing digits from $A \subseteq \{1,\dots,m-1\}$ we define

$$\mathcal{D}_{\text{c.log}}[A] := \{x \in [0,1] | d_i(x) \in A\}.$$

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Let $[(d_1,\dots,d_n)](x)=\log_m(d_n+\log_m(d_{n-1}+\dots+\log_m(d_1+x).).$

Theorem

$$L_n \le \dim_H \mathcal{D}_{c.log}[A] \le U_n$$

for all $n\ge 1$, where U_n and O_n are given by

$$\sum_{d_1,\dots,d_n\in A}[(d_k)]'(1)^{U_n}=1 \qquad \sum_{d_1,\dots,d_n\in A}[(d_k)]'(0)^{L_n}=1$$

For $m=4$ using Mathematica we get

$$\dim_H \mathcal{D}_{\mathrm{c.log}}[\{1,2\}=0.81\pm 0.01$$

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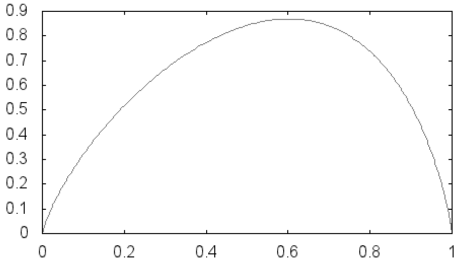
For an arbitrary continued logarithm expansion to base $m \geq 3$ we consider the set of real numbers $\mathcal{F}_{c.log}[\mathbf{p}]$ with frequency of digits given by by a probability vector $\mathbf{p}$.

Theorem

$$\dim_H \mathcal{F}_{c.log}[\mathbf{p}] \leq c < 1$$

for all $\mathbf{p}$ *(!).*

For $m = 3$ the upper bound look as follows



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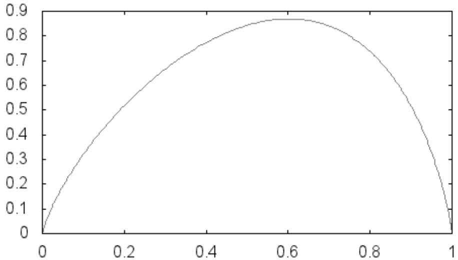

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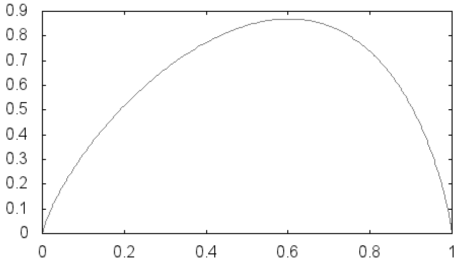

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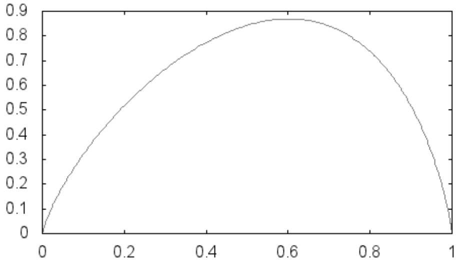

Thanks for Your Attention